Derivatives (Comm 4202)
Assignment #3 – Solution

1. It is March 5, 2020. A stock index is traded at a level of 3123, the risk-free rate is 2.53% with continuous compounding, and the dividend yield is 3.08%. What is the futures price for an index futures contract that expires on September 18, 2020?

   Answer:

   $$F = e^{(r-q)T} = Se^{(0.0253-0.0308)197/365} = 3113.74.$$

2. The USD-CAD exchange rate is 1.3283, the 6-month Canadian risk-free rate is 1.73%, and the 6-month U.S. risk-free rate is 0.8%. Both risk-free rates are continuously compounded. From a U.S. investor's perspective, what is the futures price for a USD-CAD currency futures contract that expires in six months?

   Answer:
   The futures price in CAD is is

   $$F = 1.3283e^{(0.0173-0.008)*0.5} = 1.3349.$$

   Thus, in USD, the futures price is $1/1.3345 = 0.7493$.

3. A stock is expected to pay a dividend of $1 per share in two months and in five months. The stock price is currently traded at $50 per share, and the 2-month, 3-month, 5-month, and 6-month risk-free rates are 3%, 3.025%, 3.05%, and 3.10% per annum with continuous compounding, respectively.

   (a) What is the 6-month futures price on the stock? What is the initial value of the futures contract to a long position holder?

   Answer:

   $$I = 1 * e^{-0.03*2/12} + 18e^{-0.03025*5/12} = 1.9825.$$

   Thus, the futures price is

   $$F = (S - I)e^{0.0310*6/12} = 48.7676.$$

   The initial value of the futures contract to a long position holder is 0.

(b) If the price of the stock is \$48 and the term structure of interest rates is unchanged in three months, what will be the futures price and the value of the short position in the futures contract?

Answer:

$$I_1 = 1 * e^{-0.03*2/12} = 0.9950.$$

and the value to the short position holder is

$$f = -(S_1 - I_1 - Ke^{-r*T} = -(48 - 0.9950 - 48.7656e^{-0.03025*3/12}) = 1.3952.$$

4. A stock price is currently \$40. Over each of the next two three-month periods it is expected to go up by 10% or down by 10%. The risk-free interest rate is 5% per annum with continuous compounding.

(a) What is the value of a six-month European call option with a strike price of \$42?

Answer:

$$p = \frac{e^{0.05*0.25} - 0.9}{1.1 - 0.9} = 0.5629.$$
$$f_{uu} = 6.4, \quad f_{ud} = 0, \quad f_{dd} = 0.$$

Thus,

$$\begin{cases} f_u &= e^{-0.05*0.25}(6.4 * 0.5629 + 0.4371 * 0) = 3.5578. \\ f_d &= e^{-0.05*0.25}(0 * 0.5629 + 0.4371 * 0) = 0. \end{cases}$$

and the price of the call is

$$f = e^{-0.05*0.25}(0.5629 * 3.5578 + 0.4371 * 0) = 1.9778$$

(b) What is the value of a six-month American put option with a strike price of \$42?

Answer:

If the option is a put

$$g_{uu} = 0, \quad g_{ud} = 2.4, \quad g_{dd} = 9.6$$

Thus,

$$\begin{cases} g_u &= e^{-0.05*0.25}(0 * 0.5629 + 0.4371 * 2.4) = 1.0360. \\ g_d &= e^{-0.05*0.25}(2.4 * 0.5629 + 0.4371 * 9.6) = 5.4783. \end{cases}$$

and the price of the put is

$$g = e^{-0.05*0.25}(0.5629 * 1.0360 + 0.4371 * 5.4783) = 2.9408.$$

5. Consider an option on a stock when the stock price is \$30, the exercise price is \$29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

   (a) What is the price of the option if it is a European call or put based on the Black-Scholes-Merton model?

   Answer:

   $$d = \frac{\ln(S/K) + (r - q + 1/2\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln(30/29) + (0.05 - 0 + 0.5 * 0.25^2) * (4/12)}{0.25 * \sqrt{4/12}} =$$

   If the option is a European call,

   $$call_e = 30*N(0.4225)-29*e^{-0.05*4/12}*N(0.4225-0.25*\sqrt{4/12}) = 2.5251.$$

   If the option is a European put, the value of the option is

   $$put_e = 29*e^{-0.05*4/12}*N(-0.4225+0.25*\sqrt{4/12})-30*N(-0.4225) = 1.0458.$$

   (b) What is the price of the option if it is an American call or put based on a two-period binomial model?

   Answer:

   $$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25*\sqrt{2/12}} = 1.1075, \quad d = 1/u = 0.9030.$$

   Thus,

   $$p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.05*2/12} - 0.9030}{1.1075 - 0.9030} = 0.6824.$$

   If the option is an American call,

   $$f_{uu} = 30*1.1075^2-29 = 7.7967, \quad f_{ud} = f_{du} = 30*1.1075*0.9030-29 = 1.0022, \quad f_{dd}$$

   $$\begin{cases} f_{u,\ exercise} & = 30 * 1.1075 - 29 = 4.2250 \\ f_{u,\ hold} & = e^{-0.05*2/12}(0.6824 * 7.7967 + 0.3176 * 1.0022) = 5.5902. \end{cases}$$

   $$\begin{cases} f_{d,\ exercise} & = 30 * 0.9030 - 29 = -1.9100 \\ f_{d,\ hold} & = e^{-0.05*2/12}(0.6824 * 1.0022 + 0.3176 * 0) = 0.6782. \end{cases}$$

Hence, the option should not be exercised, and the value of the call is

$$call_a = e^{-0.05*2/12} * (0.6824 * 5.5902 + 0.3158 * 0.6782) = 3.9955$$

(Note the differences of model prices of the call options. In theory, European and American call options with same maturity and strike price must have the same value, if the underlying stock does not pay dividends. However, the call option price under the BSM is much smaller than that under the two-period binomial model.)

Now, if the option is an American put,

$$g_{uu} = 0, \quad g_{ud} = g_{du} = 0, \quad g_{dd} = \max(29 - 30 * 0.9030^2, 0) = 4.5377.$$

$$\begin{cases} g_{u,\ exercise} & = 29 - 30 * 1.1075 = -4.2250 \\ g_{u,\ hold} & = e^{-0.05*2/12}(0.6824 * 0 + 0.3176 * 0) = 0. \end{cases}$$

$$\begin{cases} g_{d,\ exercise} & = 29 - 30 * 0.9030 = 1.9100 \\ g_{d,\ hold} & = e^{-0.05*2/12}(0.6824 * 0 + 0.3176 * 4.5377) = 1.4211. \end{cases}$$

That is, the put should be exercised on the downturn in period 1, and the value of the put is

$$put_a = e^{-0.05*2/12} * (0.6824 * 0 + 0.3158 * 1.9100) = 0.5982.$$

(Note that, while in theory American puts are more valuable that the European counterparts, the American put option price under the two-period binomial model is much less than the European counterpart under the BSM model.)