1. Let X have a Poisson distribution with probability mass function

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, \dots .$$

i) Find the moment generating function of X.

ii) Let $X_1, \dots, X_n$ be independent Poisson random variables, with X_i having parameter λ_i, $i = 1, \dots, n$, and let $Y = X_1 + \dots + X_n$. Find the probability distribution of Y and find its mean and variance. [6 marks]

iii) Suppose now that X_i has a Poisson distribution with parameter θi, $i = 1, \dots, n$. Find $\hat{\theta}$, the maximum likelihood estimator of θ given an independent sample $X_1, \dots, X_n$.

iv) Find the mean and variance of $\hat{\theta}$. Hence or otherwise determine whether or not $\hat{\theta}$ is consistent as $n \to \infty$.

2. (a) Consider the linear model

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2(4 - 6X_i^2) + \varepsilon_i, \quad i = 1, \dots, n.$$

We take $n = 3$ observations y_1, y_2 and y_3 where $x_1 = -1$, $x_2 = 0$ and $x_3 = 1$.

i) Find the least squares estimates of β_0, β_1 and β_2.

ii) If we consider the simpler model

$$Y_i = \beta_0 + \beta_2(4 - 6X_i^2) + \varepsilon_i, \quad i = 1, \dots, n,$$

show that the least squares estimates of β_0 and β_2 are unchanged.

(b) In a model selection procedure with $n = 10$ observations and 2 possible explanatory variables, the best models with 1 and 2 variables, respectively, are

$$M_1 : Y_i = \beta_0 + \beta_1 X_{1i} + \varepsilon_i \qquad R^2 = 0.512,$$
$$M_2 : Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \varepsilon_i \qquad R^2 = 0.701.$$

We assume that for $i = 1, \dots, n$, $\varepsilon_i \sim N(0, \sigma^2)$ are independent. The analysis of variance (ANOVA) table for model M_1 is

Source	Df	Sum Sq	Mean Sq	F
Regression	1	5.695	5.695	8.387
Residual	8	5.428	0.679	
Total	9	11.123		

i) Perform a test with significance level $\alpha = 0.05$ for the existence of regression in model M_1.

ii) Construct the ANOVA table for model M_2.

iii) Given that X_1 is included in the model, is it worth including X_2 as well? Perform a suitable hypothesis test with $\alpha = 0.05$.

3. (a) X and Y are continuous random variables with joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} 3x/4 & 0 < x < 2, 0 < y < 2-x, \\ 0 & \text{otherwise.} \end{cases}$$

i) Find $P(X > 1)$.

ii) Find the marginal distribution functions of X and Y and determine whether X and Y are independent or not.

(b) Suppose $\boldsymbol{X} = (X_1, X_2, X_3)^T \sim N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

i) Find the joint distribution of the random variables $Y_1 = X_1$ and $Y_2 = X_2 + 2X_3$.

ii) Determine whether Y_1 and Y_2 are independent or not.

iii) Find $E[Y_1 Y_2]$.

4. (a) Recall that if $X \sim Binomial(n, p)$ then, for large n, it is approximately the case that $X \sim N(np, np(1-p))$.

A pharmaceutical company tests a drug on a random sample of 2400 people. The drug was found to be effective in 1440 people.

i) Use the normal approximation to the Binomial distribution to construct an approximate 95% confidence interval for the proportion of people for whom the drug is effective.

ii) Assuming that the proportion of people who found the drug to be effective remains the same, how many people would the pharmaceutical company need to test in order for the confidence interval to have width less than 0.02?

(b) A population-wide study shows that the mean systolic blood pressure for adult males is 128 with population standard deviation 15. A group of doctors is concerned that the blood pressure of male executives is higher than average. A sample of 72 male executives finds that their mean blood pressure is 130.

i) Specify the null and alternative hypotheses which the doctors are interested in testing.

ii) Test the hypotheses using a significance level of $\alpha = 0.05$.

iii) Calculate the p-value of the test.

5. Let $X_1, \ldots, X_n$ be a random sample from the probability distribution with probability density function

$$f(x) = \theta x^{\theta-1}, \quad 0 < x < 1,$$

where $\theta > 0$.

i) Find $\hat{\theta}_{mom}$, the method-of-moments estimator of θ.

ii) Show that $\hat{\theta}_{mom}$ is biased if $n = 1$ and $\theta = 1$.

Consider estimation for θ using Bayesian inference. Suppose that θ has prior probability density function

$$\pi(\theta) = \lambda e^{-\lambda\theta}, \quad \theta > 0,$$

where $\lambda > 0$ is a fixed known value.

iii) Find the posterior density of θ given a random sample $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$.

iv) Suppose now that n is large. Which of the following two situations gives the most precise posterior inference about θ, and why?
Situation A: $x_1, \ldots, x_n$ are all very close to 0.
Situation B: $x_1, \ldots, x_n$ are all very close to 1.

[Useful information: If X has a Gamma distribution with parameters α, β then X has probability density function

$$f(x) \propto x^{\alpha-1} e^{-\beta x}, \quad x > 0,$$

and X has mean $\alpha\beta^{-1}$ and variance $\alpha\beta^{-2}$.]

6. (a) A study was carried out in which the mathematical ability of 31 children was measured before and after six months of music lessons. A second group of 26 children had computer programming lessons and their mathematical ability was also tested. The children who took music lessons had a mean improvement in mathematical ability of $\bar{x}_1 = 3.62$ with sample variance $s_1^2 = 9.33$. The corresponding results for the children who took computer programming lessons were $\bar{x}_2 = 1.39$ and $s_2^2 = 5.87$.

i) Using significance level $\alpha = 0.05$, test whether or not there exists a difference in the improvement of mathematical ability between the two groups of children. (You may need to use two different hypothesis tests.)

ii) State the assumptions you made in order for your results to be valid.

(b) Two volleyball teams, A and B, play a game. Team A wins 70% of points when they serve. However, they only win 40% of points when team B serves. The team that wins a point serves on the next point. Let X_n denote the team that serves on the nth point of the game. Then $\{X_n : n = 1, 2, \ldots\}$ is a Markov chain on the state space $S = \{A, B\}$.

i) Write down the transition matrix of the Markov chain.

ii) If team A serves on point 11 of the game, what is the probability that they also serve on point 15 of the game?

iii) A fair coin is tossed to decide which team will serve first. What is the probability that team A serves on point 5 of the game?