Inversion Theory
Assignment 4

Due Date : 10/07/2020

1. Contraction operator

An operator C of X into itself is called a contraction operator if there exists a positive real number $r < 1$ with the property that $\mu(C(x),\ C(x_0)) \leq r\mu(x, x_0)$ for all points x and x_0 in X.

Prove that the contraction operator is a continuous operator.

2. Bounded operator

Linear operator $\boldsymbol{y} = L(\boldsymbol{x})$ is called bounded if there exists a real number M with the property that

$$||L(x)|| \leq M||x|| \qquad (1)$$

for every $x \in X$.

Prove that the linear operator L is continuous if it is bounded.

3. Gram - Schmidt orthogonalization process

It is easier solving a system of linear equation if use an orthonormal set of basis elements. The set of elements $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3, \cdots, \boldsymbol{d}_n\}$ can be converted into a corresponding orthonormal set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \cdots, \boldsymbol{e}_n\}$ as follws.

The first step is

$$\boldsymbol{e}_1 = \frac{\boldsymbol{d}_1}{||\boldsymbol{d}_1||} \qquad (2)$$

The e_2 and e_3 are calculated as follows:

$$\boldsymbol{e}_2 = \frac{\boldsymbol{d}_2 - (\boldsymbol{d}_2, \boldsymbol{e}_1)\boldsymbol{e}_1}{\|\boldsymbol{d}_2 - (\boldsymbol{d}_2, \boldsymbol{e}_1)\boldsymbol{e}_1\|} \tag{3}$$

$$\boldsymbol{e}_3 = \frac{\boldsymbol{d}_3 - (\boldsymbol{d}_3, \boldsymbol{e}_2)\boldsymbol{e}_2 - (\boldsymbol{d}_3, \boldsymbol{e}_1)\boldsymbol{e}_1}{\|\boldsymbol{d}_3 - (\boldsymbol{d}_3, \boldsymbol{e}_2)\boldsymbol{e}_2 - (\boldsymbol{d}_3, \boldsymbol{e}_1)\boldsymbol{e}_1\|} \tag{4}$$

The tasks: a) proof that $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ form an orthonormal set of elements

b) present similar expressions for $\boldsymbol{e}_4$ and $\boldsymbol{e}_5$.

4. 2-D Gravity Inverse problem

Let us consider a 2-D gravity problem. The anomalous gravity field g_z is generated by some mass distributed in rectangular domain S within the lower half plane with the unknown excess density $\Delta\rho(x,z)$. The gravity field observed on the earth's surface is given (same as in HW3).

Problem:

Solve the inverse gravity problem of determining the excess density distribution, $\Delta\rho(x,z)$, using Riesz representation theorem.

Directions:

The anomalous 2-D gravity field can be calculated by the formula:

$$g_z(x',0) = 2\gamma \iint_S \frac{\rho(x,z)\cdot z}{(x-x')^2+z^2}\,dzdx \qquad (5)$$

We have observations at 41 points, which can be written as follows:

$$d_j = g_z(x_{j\,,\,0}) = 2\gamma \iint_{S_i} \frac{\rho(x,z)\cdot z}{(x-x_j)^2+z^2}\,dzdx =$$

$$= \iint_S \rho(x,z)\left\{2\gamma \frac{z}{(x-x_j)^2+z^2}\right\}dzdx \qquad (6)$$

According to Riesz representation theorem, there exist the vectors $l^{(j)}$ (the elements of the model space M), which can be used to represent the observed data:

$$d_j = \left(\rho(x,z), l^{(j)}\right)$$
$$j = 1, 2, \cdots n;\ l^{(j)} \in M. \quad (7)$$

Vectors $l^{(j)}$ are called “the data kernels”. Comparing (7) and (6), we see that

$$l^{(j)} = l^{(j)}(x,z) = 2\gamma \frac{z}{\left(x - x_j\right)^2 + z^2} \quad (8)$$

The unknown distribution of the density, according to the Riesz representation theorem, can be expressed as

$$\rho(x,z) = \sum_{j=1}^{n} \beta_j l^{(j)} = 2\gamma \sum_{j=1}^{n} \beta_j \frac{z}{\left(x - x_j\right)^2 + z^2} \quad (9)$$

where the coefficients β_j satisfy the following system of linear equations:

$$d_i = g_z(x_j, 0) = \sum_{j=1}^{n} \Gamma_{ij} \beta_j \quad (10)$$

We can calculate the Gram matrix as follows:

$$\Gamma_{ji} = \left(l^{(i)}, l^{(j)}\right) = 4\gamma^2 \iint_S \frac{z^2}{[(x - x_i)^2 + z^2]\left[\left(x - x_j\right)^2 + z^2\right]} dzdx \tag{11}$$

$$j = 1, 2, \cdots, n; \;\; i = 1, 2, \cdots, n.$$

You are recommended to use Matlab for this assignment. Assume that resulting density image consists of a number of small enough pixels. Take 100 cells in x direction and 50 cells in z direction. The image has to occupy a rectangular subsurface region (0,1000) along the X axis and (25,225) along the Z axis.

Note that in this approach, we approximate one data kernel $l^{(j)}(x_k, z_l)$ by a matrix (100*50). The density image, $\rho(x_k, z_l)$, will have the same dimensions as the data kernel, and it will be a matrix (100*50). One can reshape these matrices into the row vectors $l^{(j)}$ and ρ of the length 5000. You can introduce now a matrix **F**, with the rows, formed by the row vectors $l^{(j)}$. The Gram matrix consists of the combination of the dot products (a sum over products of scalar components of elements) of the data kernels, which can be written in matrix notations as

$$\boldsymbol{\Gamma} = \boldsymbol{F} \cdot \boldsymbol{F}^T \qquad (12)$$

The linear system of equations (10) takes the matrix form:

$$\boldsymbol{\Gamma} \cdot \boldsymbol{\beta} = \boldsymbol{d} \qquad (13)$$

where β is the vector of unknown coefficients, and d is the vector of the observed data. Solve system (13) using MATLAB.

Now, you can find the density distribution as a linear combination of the data kernels with the coefficients β_i, which can be written in matrix form as

$$\rho = F^T \beta \qquad (14)$$

where ρ is the vector of the density distribution. For visualization, this vector can be reshaped back to matrix (100*50) of $\rho(x_k, z_l)$.

Find the gravity field produced by this density distribution (in other words, solve the forward modeling problem). Input the density distribution and the predicted field into Matlab and view it using your own visualization routine or the one provided in the previous assignment. The data files and visualization scripts will be e-mailed to you.

Note that, the density image does not look like the image from the previous assignment (hw3), but produces the same data. These two solutions (from this assignment and assignment #3) are called the "equivalent solutions".

Turn in the plots of the obtained density image, predicted field values, and the program listing.

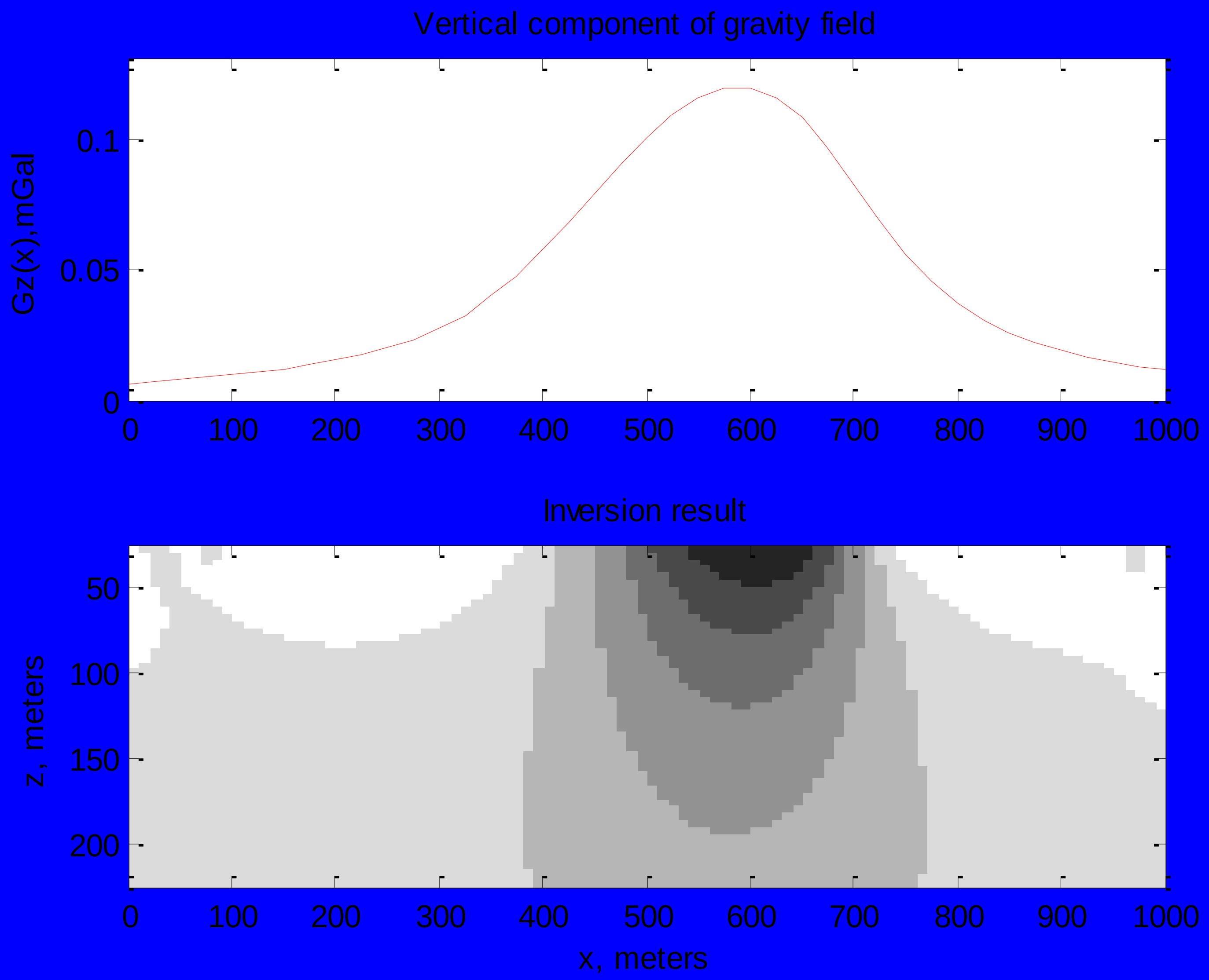
$$\boldsymbol{F} = \begin{bmatrix} l_1^{(1)} & l_1^{(2)} & \cdots & l_1^{(5000)} \\ l_2^{(1)} & l_2^{(2)} & \cdots & l_2^{(5000)} \\ \vdots & \vdots & & \vdots \\ l_{41}^{(1)} & l_{41}^{(2)} & \cdots & l_{41}^{(5000)} \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{41} \end{bmatrix}, \boldsymbol{d}^0 = \begin{bmatrix} d^0{}_1 \\ d^0{}_2 \\ \vdots \\ d^0{}_{41} \end{bmatrix}$$

$$l_i(x') = 2\gamma \frac{z}{(x - x')^2 + z^2}$$

$$\boldsymbol{\Gamma} = \boldsymbol{F} \cdot \boldsymbol{F}^T \ \ (41 \times 41)$$

$$\boldsymbol{\Gamma\beta} = \boldsymbol{d}^0$$

$$\boldsymbol{\rho} = \boldsymbol{F}^T \cdot \boldsymbol{\beta}$$



Vertical component of gravity field
Gz(x),mGal
0.1
0.05
0
0
100
200
300
400
500
600
700
800
900
1000
Inversion result
z, meters
50
100
150
200
0
100
200
300
400
500
600
700
800
900
1000
x, meters